On Proving Parameterized Size Lower Bounds for Multilinear Algebraic Models

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Abstract. We consider the problem of obtaining parameterized lower bounds for the size of arithmetic circuits computing polynomials with the degree of the polynomial as the parameter. In particular, we consider the following special classes of multilinear algebraic branching programs: 1) Read Once Oblivious Algebraic Branching Programs (ROABPs); 2) Strict interval branching programs; and 3) Sum of read once formulas with restricted ordering.
We obtain parameterized lower bounds (i.e., $n^{\Omega(t(k))}$ lower bound for some function t of k) on the size of the above models computing a multilinear polynomial that can be computed by a depth four circuit of size $g(k)n^{O(1)}$ for some computable function g.
Our proof is an adaptation of the existing techniques to the parameterized setting. The main challenge we address is the construction of hard parameterized polynomials. In fact, we show that there are polynomials computed by depth four circuits of small size (in the parameterized sense), but have high rank of the partial derivative matrix.

1 Introduction

Parameterized Complexity is a multi-dimensional study of computational problems which views the complexity of a problem in terms of both the input size and an additional parameter. This leads to a finer classification of computational problems, and a relaxed notion of tractability, given by a $f(k)\text{poly}(n)$ bound on the running time for decision problems with parameter k, known as *fixed-parameter tractability* or FPT. This was first studied by Downey and Fellows in their seminal work [8] where they developed parameterized complexity theory, and paved the way for extensive study of parameterized algorithms. The notion of intractability in parameterized complexity is captured by the W-hierarchy of classes [8].

Algebraic Complexity Theory is concerned with complexity of computing polynomials using elementary arithmetic operations such as addition and multiplication over an underlying ring or field. Valiant [25] formalized the notions of algebraic complexity theory and posed proving lower bound on the size of arithmetic circuits computing explicit polynomials as the primary challenge for the area. Following Valiant's work, there has been intense research efforts in the past four decades to prove lower bounds on the size of special classes of arithmetic circuits such as constant depth circuits, multilinear formula and non-commutative models. (See [23, 24] for a survey.) Despite several techniques, the best known size lower bound for general arithmetic circuit is only super linear [3].

Given the lack of progress towards proving lower bounds on arithmetic circuits and the success of parameterized complexity theory in refining the notion of tractability, it is worthwhile exploring the feasibility of parameterizations of polynomials. Engels [9] initiated development of a parameterized theory for algebraic complexity classes and suggested suitable notions of tractability and reductions. The attempt in [9] was more at obtaining a complexity classification in the form of complete problems for a generic parameter. Müller [18] was the first to introduce parameterizations on polynomials in the context of designing parameterized algorithms problems on polynomials such as testing for identity of polynomials given as arithmetic circuits (ACIT). ACIT is one of the fundamental problems in algebraic complexity theory and has close connections to the circuit lower bound problem [15]. Müller studied parameters such as the number of variables in the polynomial, multiplication depth of the circuit computing the polynomial etc and obtained efficient randomized parameterized algorithms for ACIT. It may be noted that ACIT is non-trivial with these parameters since a polynomial can potentially have $n^{\Omega(k)}$ monomials where k is any of these parameters. In [6] Chauhan and Rao studied ACIT with degree as the parameter and obtained a randomness efficient parameterized algorithm for ACIT. It may be noted that polynomials with the degree bounded by the parameter are widely used in developing efficient parameterized algorithms [4, 1, 10] and in expressing properties of graphs [5]. For example, in [10] the polynomial representing homomorphisms between two graphs has indeed degree equal to the parameter, i.e. the number of vertices in the pattern graph. It may be noted that efficient computation of the polynomial defined in [10] by arithmetic formulas leads to space-efficient algorithms for detecting homomorphisms between graphs of bounded treewidth. In [12] the authors along with Prakash studied polynomials parameterized by the degree and showed limitations of an existing approach in obtaining deterministic parameterized algorithms for ACIT. In this article, we explore the possibility of obtaining parameterized lower bounds for the size of arithmetic circuits with degree as the parameter.

Let n be the number of variables and k be a parameter (e.g., degree of the polynomial). Throughout the article, $t(k)$ denotes a computable function that depends only on the parameter, e.g., $t(k) = 2^k, t(k) = 2^{2^k}, t(k) = \sqrt{k}$ etc. Any circuit is said to be of fpt size if the size of the circuit is bounded by $t(k)n^{O(1)}$ for some computable t. By a parameterized lower bound, we mean a lower bound of the form $n^{\Omega(t(k))}$ for a computable function t. It may be noted that the task of proving parameterized lower bounds is more challenging than classical lower bounds. In the case of degree as a parameter, most of the existing lower bounds of the form $n^{\Omega(\sqrt{k})}$(e.g., [13, 11, 17]) for constant depth circuits are already parameterized lower bounds. In contrast, the lower bounds for other special classes such as multilinear formula [21, 20] do not translate easily.

To understand the primary challenge in translating the results in [20, 21] to the parameterized setting we need to delve a bit on the techniques used for proving lower bounds. Raz [21] used the notion of partial derivative matrix under a partition of variables with equal parts (See Section 2 for a detailed definition) as a measure of complexity. The idea is to show existence of such partitions where polynomials computed by a multilinear formula of small size will have small rank for the partial derivative matrix. Then for any polynomial that has large rank under every partition, a natural lower bound on the size follows. However, the analysis done in [21] or subsequent works [22, 20, 7] do not carry forward when parameterized by the degree. Similarly the construction of the polynomial family with high rank partial derivative matrix in [21] and subsequent works do not generalize to the parameterized setting.

In this article we address the challenge of translating lower bounds for the size of multilinear restrictions of arithmetic circuits to parameterized lower bounds.

Results Our primary result is a parameterized family of polynomials (Theorem 1) such that under every equal sized bi-partition of the variables, the rank of the partial derivative matrix is the maximum possible up to a factor that depends only on the parameter. Further, we demonstrate a simple parameterized family of polynomials that can be written as a sum of three read once formulas such that under any partition of the variables into two equal parts, the rank of the partial derivative matrix is high. (Theorem 2) As a consequence, we obtain parameterized lower bounds for the size of an ROABP (Theorem 3), a strict-interval ABP (Corollary 1) and sum of ROPs with restricted ordering of variables (Theorem 6) against the constructed hard polynomial.

Finally, we obtain a parameterized version of the separation between read-3 ABPs and ROABPs given in [16] (Theorem 4). This is done by constructing a parameterized variant of the hard polynomial given in [16] (Theorem 2).

2 Preliminaries

In this section we give all basic definitions related to arithmetic circuit. Let $\mathbb{F}$ denote a field. Most of the arguments in this article work for any $\mathbb{F}$. Let $X = \{x_1, \ldots, x_n\}$ denote the set of variables.

An *arithmetic circuit* is a model for computing polynomials using the basic operations $+$ and $\times$. An arithmetic circuit C is a directed acyclic graph, where every node (called a gate) has in-degree two or zero. The gates of in-degree zero are called input gates and are labeled from $X \cup \mathbb{F}$, where $X = \{x_1, \ldots, x_n\}$ is the set of variables called inputs and $\mathbb{F}$ is the underlying field. Internal gates of C are labeled by either $+$ or $\times$. Gates of out-degree zero are called output gates. Typically an arithmetic circuit will have a single output gate. Every gate in the circuit C is associated with a unique polynomial in $\mathbb{F}[X]$. The polynomial computed by the circuit is the polynomial associated at its output gate.

The complexity of arithmetic circuits is measured in terms of *size* and *depth*. Size is defined as the number of $+$ and $\times$ operations in the circuit. Depth of the circuit represents the length of the longest path from the output node (root) to an input node (leaf) of the circuit. Since a constant depth arithmetic circuit where fan-in of every gate is bounded by 2 (or even a constant) cannot even read all of the inputs, we assume unbounded fan-in in the case of constant depth circuits. Arithmetic circuits of constant depth have received wide attention [23].

An arithmetic circuit C is said to be *syntactic multilinear* if for every product gate $f = g \times h$ in C, the set of variables that appear under the sub-circuit rooted at g is disjoint from that of h. Naturally, a syntactic multilinear circuits computes a multilinear polynomial.

An arithmetic circuit where the underlying graph is a tree is known as *arithmetic formula*. An arithmetic formula is said to be read once (ROF for short) if every variable appears as a label in at most one leaf. Polynomials computed by ROFs are known as Read-once polynomials (ROPs for short). It may also be noted that ROFs are a proper subclass of syntactic multilinear formulas.

An algebraic branching program (ABP) P is a directed acyclic graph with a source vertex s of in-degree 0 and a sink vertex t of out-degree 0. The rest of the vertices can be divided into layers $L_1, L_2, \ldots, L_{r-1}$ between s and t, s being the only vertex in L_0, the first layer, and t being the only vertex in the last layer ℓ_r. Edges in P are between vertices of consecutive layers. Every edge e is labelled by either a constant from $\mathbb{F}$ or a variable from X. For a directed path ρ in P, let $w(\rho)$ denote the product of edge labels in ρ. For any pair of nodes u, v in P let $[u, v]_P$ denote the polynomial $\sum_{\rho \text{ is a } u \to v \text{ path}} w(\rho)$. The polynomial computed by P is $[s, t]_P$. The size of an ABP is the total number of nodes and edges in it and the depth of an ABP is the total number of layers in it excluding the layers containing s and t. Read-once ABPs are such that every input variable is read at most once along any path from s to t. Read-once Oblivious ABPs (ROABPs) are such that input variables are read at most once, in a fixed order, along any path from s to t, and any variable occur as a label in at most one layer of the program.

Let π be a permutation of the variables. An interval in π is a set of the form $\{\pi(i), \pi(i+1), \ldots, \pi(j)\}$ for some $i < j$. Arvind and Raja [2] studied a restriction of multilinear ABPs called as interval ABPs where every node in the ABP computes a polynomial whose variable set forms an interval in $\{1, \ldots, n\}$. In this article, we consider a restriction of interval ABPs which we call as *strict interval* ABPs. A syntactically multilinear ABP P is said to be a π strict interval ABP, if for any pair of nodes (a, b) in P, the index set X_{ab} of the variables occurring on all paths from a to b is contained in some π interval I_{ab} in $[n]$ and for any node c the intervals I_{ab} and I_{bc} are non-overlapping.

For a polynomial $p \in \mathbb{X}$, let $\mathsf{var}(p)$ denote the set of variables that p is dependent on and $\deg(p)$ denote its degree.

Partial Derivative Matrix of a polynomial Nisan [19] defined the partial derivative matrix of a polynomial and considered its rank as a complexity measure for non-commutative polynomials and proved exponential lower bounds for the size of non-commutative formulas. Raz [21] considered a variant of the partial derivative matrix and proved super polynomial size lower bounds for multilinear formulas. We describe the partial derivative matrix introduced by Raz [21] in more detail. Let $X = \{x_1, \ldots, x_n\}$ be the set of variables where n is even. A partition of X is an injective function $\varphi : X \to Y \cup Z$, where Y and Z are two disjoint sets of variables. A partition φ is said to be an equi-partition if $|Y| = |Z| = n/2$. In the remainder of the article, we assume that the number of variables n is an even number.

Definition 1. *[21] Let $f \in \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial of degree d, $\varphi : X \to Y \cup Z$ be a partition of the input variables of f. Then the partial derivative matrix of f with respect to φ, denoted by M_{f^φ} is a $2^{|Y|} \times 2^{|Z|}$ matrix where the rows are indexed by the set of all multilinear monomials μ in the variables Y, and columns indexed by the set of all multilinear monomials ν in variables in Z. For monomials μ and ν respectively in variables Y and Z, the entry $M_{f^\varphi}(\mu, \nu)$ is the coefficient of the monomial $\mu\nu$ in f.*

For a multilinear polynomial $p \in \mathbb{F}[X]$ and an equi-partition φ, let $\mathsf{rank}_\varphi(p)$ be the rank of the matrix M_{p^φ} over $\mathbb{F}$. The following fundamental properties of the rank of a partial derivative matrix was given by Raz [21].

Lemma 1. *[21] Let f_1 and f_2 be multilinear polynomials. Then $\mathsf{rank}_\varphi(f_1 + f_2) \leq \mathsf{rank}_\varphi(f_1) + \mathsf{rank}_\varphi(f_2)$ and if $\mathsf{var}(f_1) \cap \mathsf{var}(f_2) = \emptyset$ then $\mathsf{rank}_\varphi(f_1 f_2) = \mathsf{rank}_\varphi(f_1)\mathsf{rank}_\varphi(f_2)$.*

Lemma 2. *For any equi-partition $\varphi : X \to Y \cup Z$, and any multilinear polynomial p of degree k, we have $\mathsf{rank}_\varphi(p) \leq (k/2 + 1)\binom{n/2}{k/2}$.*

3 Construction of hard parameterized polynomials

This section is devoted to the construction of two parameterized polynomial families $f = (f_{n,2k})_{k \geq 0}$ and $h = (h_{2n,k})$. The first family is computable by a depth four circuit of fpt (i.e., $t(k)n^{O(1)}$ for a computable t) size and the second family is a sum of three ROFs. Further, for any partition φ, $\mathsf{rank}_\varphi(f)$ is the maximum possible value up to a factor that depends only on the parameter and $\mathsf{rank}_\varphi(h)$ is the maximum possible value up to a constant factor in the exponent.

A full rank polynomial: It may be noted that for a multilinear polynomial g of degree k in n variables, the maximum possible value of $\mathsf{rank}_\varphi(g)$ over all partitions φ is at most $(k/2+1)\binom{n/2}{k/2}$. Though it is possible to construct polynomials that achieve this bound under a fixed partition φ, it is not immediate if there is a polynomial g computed by small circuits that is full rank under every equipartition. In the following, we give the description of a multilinear polynomial of degree k that has rank $n^{k/2}/t(k)$ where t is a function that depends only on k. We assume that $2k|n$. Suppose $V_1 \cup \cdots \cup V_{2k} = X$ be a partition of the variable set $X = \{x_1, \ldots, x_n\}$ such that $|V_i| = |V_j|$ for $1 \leq i < j \leq 2k$.

For convenience let $V_i = \{x_{i,1}, \ldots, x_{i,n/2k}\}$, where we assume a natural ordering among the variables.

Let $\mathcal{M}$ be the set of all possible perfect matchings on $G = K_{2k}$, the complete graph on $2k$ vertices. Let ζ_M for $M \in \mathcal{M}$, $\omega_{i,j}$ $1 \le i < j \le n/k$ be formal variables. Let $\mathbb{G}$ be any extension of $\mathbb{F}$ containing $\{\omega_{i,j}\} \cup \{\zeta_M \mid M \in \mathcal{M}\}$. We define a parameterized family of polynomial $f = (f_{n,2k})$, $f_{n,2k} \in \mathbb{G}[x_1, x_2, \ldots, x_n]$ as follows:

$$f(x_1, x_2, \ldots, x_n) = \sum_{M \in \mathcal{M}} \zeta_M \prod_{(i,j) \sim M} (1 + p(V_i \cup V_j)),$$

where p is a n/k variate quadratic multilinear polynomial defined as

$$p(v_1, \ldots, v_{n/k}) = \sum_{k<\ell} \omega_{k,\ell} v_k v_\ell.$$

Note that $f_{n,2k}$ is a degree $2k$ polynomial in n variables. When n and k are clear from the context, we use f to denote $f_{n,2k}$. Let $\mathbb{G} = \mathbb{F}(\{\zeta_M \mid M \in \mathcal{M}\} \cup \{\omega_{i,j} \mid 1 \le i < j \le n/k\})$, i.e the rational function field of the polynomial ring $\mathbb{F}[\{\zeta_M \mid M \in \mathcal{M}\} \cup \{\omega_{i,j} \mid 1 \le i < j \le n/k\}]$. In the remainder of the section, we argue that the polynomial family f defined above has almost full rank under every partition $\varphi : X \to Y \cup Z$, such that $|Y| = |Z| = |X|/2$.

Definition 2. *Consider a partition function $\varphi : X \to Y \cup Z$ such that $|Y| = |Z|$ and $V \subseteq X$. The set V is said to be ℓ-unbalanced with respect to φ if $\frac{|X|}{2} - |\varphi(X) \cap Z| = \ell = |\varphi(X) \cap Y| - \frac{|X|}{2}$.*

It may be noted that ℓ can be positive or negative accordingly as $|\varphi(X) \cap Y| > |\varphi(X) \cap Z|$ or otherwise. Our first observation is, even if the set $V = V_i \cup V_j$ is ℓ-unbalanced for $\ell < n/4k$, $i \le q \le [2k]$, $\mathsf{rank}_\varphi(p(V_i, V_j))$ remains large:

Lemma 3. *If $V_i \cup V_j$ is ℓ unbalanced with respect to a partition $\varphi : X \to Y \cup Z$, then $\mathsf{rank}_\varphi(p(V_i, V_j)) = \Omega(n/2k - |\ell|)$.*

Proof. Without loss of generality, suppose that $\ell > 0$, $V_i \cup V_j = \{v_1, \ldots, v_{n/k}\}$ and

$$\varphi(v_i) = \begin{cases} y_i & \text{if } i \le n/2k + \ell \\ z_{i-(n/2k+\ell)} & \text{otherwise.} \end{cases}$$

Since p is a quadratic polynomial, the rows of $M_{p(V_i,V_j)^\varphi}$ are indexed by monomials $\emptyset, y_1, \ldots y_{n/2k+\ell}, y_i y_j, 1 \le i < j \le n/2k+\ell$ and columns are indexed by $\emptyset, z_1, \ldots, z_{n/2k-\ell}$, $z_i z_j, 1 \le i < j \le n/2k - \ell$. The rows and columns indexed by degree 2 monomials will have a rank of at most 2. Thus it is required to show that the submatrix of $M_{p(V_i,V_j)^\varphi}$ with rows indexed by $\emptyset, y_1, \ldots y_{n/2k+\ell}$ and columns indexed by $\emptyset, z_1, \ldots, z_{n/2k-\ell}$ has rank $\Omega(n/2k - |\ell|)$.

The $(y_i, z_j)^{\text{th}}$ entry of $M_{p(V_i,V_j)^\varphi}$ contains $\omega_{i,n/2k+j}$. By suitably substituting the variables $\omega_{i,n/2k+j}$ with values from $\mathbb{F}$, we see that the submatrix of $M_{p(V_i,V_j)^\varphi}$ restricted to rows and columns indexed respectively by $\emptyset, y_1, \ldots y_{n/2k+\ell}$ and $\emptyset, z_1, \ldots, z_{n/2k-\ell}$ has rank $\Omega(n/2k - |\ell|)$. ◻

Theorem 1. *For the parameterized polynomial family $f = (f_{n,2k})_{n,k \ge 0}$ as above,*

$$\mathsf{rank}_\varphi(f_{n,2k}) = \Omega(\frac{n^k}{(2k)^{2k}})$$

for every equi-partition $\varphi : X \to Y \cup Z$ and $k > 3$.

Proof. Let φ be an equi-partition of X. Note that by the definition of f, it is enough to show that for all equi-partitions φ, there exists an optimal matching N such that $\mathsf{rank}_\varphi(f_N) = \Omega(\frac{n^k}{(2k)^{2k}})$, where $f_N = \prod_{(i,j)\in N} p(V_i, V_j)$. Since f_N is multilinear, it is enough to prove that $\forall (i,j) \in N$, $\mathsf{rank}_\varphi(p(V_i, V_j)) = \Omega(\frac{n}{k^2})$. Our argument is an iterative construction of the required matching.

We begin with some notations. Let $D(V_i) = |\varphi(V_i) \cap Y| - \frac{|V_i|}{2}$. Let $Y_i = \varphi(V_i) \cap Y$, $Z_i = \varphi(V_i) \cap Z$. We know $\forall i \in [2k]$, $|V_i| = \frac{n}{2k}$. So, $D(V_i) = |Y_i| - \frac{n}{4k}$ is the imbalance of φ. Since $0 \le |Y_i| \le |V_i| = \frac{n}{2k}$, $D(V_i) \in [\frac{-n}{4k}, \frac{n}{4k}]$.

Let $M \in \mathcal{M}$. For each edge $e = (i,j)$ in the matching M, we associate a weight with respect to φ: $\mathsf{wt}(e) = |D(V_i) + D(V_j)|$. The weight of the matching M, denoted by $\mathsf{wt}(M)$, is the sum of the weights of the edges in M, i.e., $\mathsf{wt}(M) = \sum_{e\in M} \mathsf{wt}(e)$. In the following, we give an iterative procedure, that given a matching M produces a matching N with the required properties. The procedure in each iteration, obtains a new matching of smaller weight than the given matching. The crucial observation then is, matchings that are weight optimal with respect to the procedure outlined below indeed have the required property. We say that a matching M is *good* with respect to φ, if $\forall\ e = (i,j) \in M$, $\mathsf{wt}(e) \le n/2k - n/(2k(k-1))$. Note that if M is good then for every edge $(i,j) \in M$, we have $V_i \cup V_j$ is ℓ-unbalanced for some ℓ with $|\ell| \le n/2k - n/(2k(k-1))$. Then, by Lemma 3 we have $\mathsf{rank}_\varphi(f_M) \ge n/(2k(k-1))$.

Suppose that the matching M is not good. Let $e = (i,j) \in M$ be an edge such that $\mathsf{wt}(e) > n/2k - n/2k(k-1)$. If there are multiple such edges, e is chosen such that $\mathsf{wt}(e)$ is the maximum, breaking ties arbitrarily. Note that we can assume that $D(V_i)$ and $D(V_j)$ are of the same sign, else we would have $\mathsf{wt}(e) \le n/4k$. Without loss of generality, assume that both $D(V_i)$ and $D(V_j)$ to be non-negative, i.e., $\mathsf{wt}(e) = D(V_i) + D(V_j)$. Since φ is an equi-partition, we have

$$\sum_{m\in[2k]} D(V_m) = \sum_{m\in[2k]} \left(|Y_m| - \frac{n}{4k}\right) = 0 \implies \sum_{m\in[2k]\setminus\{i,j\}} D(V_m) = -\mathsf{wt}(e)$$

$$\text{i.e., } \sum_{e'\in M\setminus\{e\}} \mathrm{sgn}(e')\mathsf{wt}(e') < \frac{-n}{2k} + \frac{n}{2k(k-1)},$$

where $\mathrm{sgn}(e)$ is ± 1 depending on the sign of $\mathsf{wt}(e)$. By averaging, there is an $e_1 \in M$ such that $\mathrm{sgn}(e_1)\mathsf{wt}(e_1) < \frac{-n}{2k(k-1)} + \frac{n}{2k(k-1)^2}$. Suppose $e_1 = (i_1, j_1)$. Let $D(V_i) = a$, $D(V_j) = b$, $D(V_{i_1}) = c$, $D(V_{j_1}) = d$. Since $c + d < 0$, it must be that either $c < 0$ or $d < 0$. The new matching is constructed based on the values of a, b, c and d.

Case 1 Suppose $c, d < 0$. Then, $|a+b|+|c+d| > |a+c|+|b+d|$. We replace the edges (i,j) and (i_1, j_1) by $(i, i_1), (j, j_1)$ to get a new matching M'. We have $\mathsf{wt}(M') < \mathsf{wt}(M)$.

Case 2 Either $c \ge 0$ and $d < 0$ or $c < 0$ and $d \ge 0$. Without loss of generality, assume that $c \ge 0$ and $d < 0$. Suppose $c > \frac{n}{4k} - \frac{n}{2k(k-1)} + \frac{n}{2k(k-1)^2}$, then we have $d < \frac{-n}{2k(k-1)} + \frac{n}{2k(k-1)^2} - c < \frac{-n}{4k}$ which is impossible as $|d| \le \frac{n}{4k}$. Therefore, we have $c \le \frac{n}{4k} - \frac{n}{2k(k-1)} + \frac{n}{2k(k-1)^2}$. If $c > a, b$, then $a + b < 2c \le \frac{n}{2k} - \frac{n}{k(k-1)} + \frac{n}{k(k-1)^2}$. For $k > 3$, this is impossible since $\mathsf{wt}(e) > \frac{n}{2k} - \frac{n}{2k(k-1)}$. We consider the following sub-cases:

Subcase (a) $a > c$. Then $a + b > c + b$, replace the edges (i,j) and (i_1, j_1) with the edges (i, j_1) and (i_1, j) to get the new matching M'.

Subcase (b) $b > c$. Then $a + b > a + c$, replace (i,j) and (i_1, j_1) with the edges (i, i_1) and (j, j_1) to get the new matching M'.

For the new matching M' obtained from M as above, we have one of the following properties:

– It has smaller total weight than M, i.e., $\mathsf{wt}(M') < \mathsf{wt}(M)$, or
– If M has a unique maximum weight edge, then the weight of any edge in M' is strictly smaller than that in M, i.e. $\max_{e' \in M'} \mathsf{wt}(e') < \mathsf{wt}(e)$, or
– The number of edges that have maximum weight in M' is strictly smaller than that in M, i.e., $|\{e'' \mid \mathsf{wt}(e'') = \max_{e' \in M'} \mathsf{wt}(e')\}| < |\{e'' \mid \mathsf{wt}(e'') = \max_{e' \in M} \mathsf{wt}(e')\}|$.

Since all of the invariants above are finite, by repeating the above procedure a finite number of times we get a matching $N \in \mathcal{M}$ such that any of the above steps are not applicable. That is, for every $e' \in N$, $\mathsf{wt}(e') \leq n/2k - n/2k(k-1)$.

Thus for every edge $(i,j) \in N$, we have $\mathsf{rank}_\varphi(p(V_i, V_j) = \Omega(n/2k(k-1))$ and $\mathsf{rank}_\varphi(f_N) = \Omega(n^k/(2k)^{2k})$. By the construction of the polynomial and Lemma 1, we have $\mathsf{rank}_\varphi(f) \geq \max_{M \in \mathcal{M}}\{\mathsf{rank}_\varphi(f_M)\} = \Omega(n^k/(2k)^{2k})$, as required. ☐

A high rank sum of three ROFs: In [16], Kayal et al. showed that there is a polynomial that can be written as sum of three ROFs such that any ROABP computing it requires exponential size. The lower bound proof in [16] is based on the construction of a polynomial using three edge disjoint perfect matchings on n vertices. We need a 3-regular mildly explicit family of expander graphs defined in [14]. Let $\mathcal{G} = (G(q))_{q>0, \text{ prime}}$ be a family of 3 regular expander graphs where a vertex x in $G(q)$ is connected to $x+1, x-1$ and x^{-1} where all of the operations are modulo q. When q is clear from the context, we denote $G(q)$ by G. Let G' be the double cover of G, i.e., $G' = (V_1, V_2, E')$ is the bipartite graph such that V_1, V_2 are copies of V and $u \in V_1, v \in V_2, (u,v) \in E' \iff (u,v) \in E$. It is known from [14] that the set of edges in E' can be viewed as the union of 3 edge disjoint perfect matchings. In [16], Kayal et al. construct a polynomial for each of these matchings and the hard polynomial is obtained by taking the sum of these three polynomials. This polynomial has degree $n/2$ and is unsuitable in the parameterized context.

We construct a polynomial h from G' similar to the one in [16], but having degree-k. Suppose $M_1 \cup M_2 \cup M_3 = E'$ be disjoint perfect matchings. We divide the n edges in each of the M_i into $\frac{k}{2}$ parts of $\frac{n}{k}$ edges each. Suppose $M_i = B_{i1} \cup B_{i2} \cup \cdots \cup B_{ik/2}$. The division is done arbitrarily. So, for each edge $(i,j) \in M$, we consider a monomial $x_i x_j$, and the final polynomial is the following:

$$h(x_1, \ldots, x_{2n}) = \sum_{i \in [3]} w_i \left(\prod_{j \in [\frac{k}{2}]} \sum_{(u,v) \in B_{ij}} x_u x_v \right),$$

where M_1, M_2 and M_3 are the edge-disjoint matchings such that $M_i = \prod_{j \in [\frac{k}{2}]} B_{ij}$, B_{ij} being the j^{th} partition of edges in the matching M_i and w_1, w_2 and w_3 are formal variables. For a partition $\varphi : X \to Y \cup Z$, and an edge $(u,v) \in M_i$, (u,v) is said to be *bichromatic* with respect to φ if either $\varphi(x_u) \in Y$ and $\varphi(x_v) \in Z$ or $\varphi(x_u) \in Z$ and $\varphi(x_v) \in Y$. For a set of edges A over $\{x_1, \ldots, x_n\}$ let $\mathsf{be}_\varphi(A)$ be the number edges in A that are bichromatic with respect to φ. For a graph $G = (V,E)$, let $\mathsf{be}_\varphi(G)$ denote $\mathsf{be}_\varphi(E)$.

Let $\mathcal{D}$ denote the uniform distribution on the set of all partitions $\varphi : X \to Y \cup Z$ such that $|Y| = |Z|$. In the following we state the desired property of the polynomial h:

Theorem 2. *Let h be the polynomial defined as above. Then there is a constant $c > 0$ such that for every equi-partition φ of X, over the rational function field $\mathbb{F}(w_1, w_2, w_3)$*

$$\mathsf{rank}_\varphi(h) \geq \left(\frac{n}{k}\right)^{ck}.$$

Proof. Let $Y \subseteq X = \{x_1, \ldots, x_n\}$, $|Y| = \frac{n}{2}$ such that $\varphi : X \to Y \cup Z$. By the expander property of G (see [16]), the number of edges from Y to Z is lower bounded by $E(Y,Z) \geq \frac{(2+10^{-4})}{2} \cdot |Y| = \frac{(1+\epsilon)n}{2}$ for a fixed $\epsilon > 0$. (See [16] for details.)

Now, each perfect matching has $\frac{n}{2}$ edges, so the graph has $\frac{3n}{2}$ edges. By averaging, we get that there is a matching M_i, $1 \leq i \leq 3$ such that the number of bichromatic edges in M_i is at least $\frac{(1+\epsilon)n}{6}$. Without loss of generality, suppose $i = 1$. Let $h_1 = \prod_{j\in[\frac{k}{2}]} \sum_{(u,v)\in B_{1j}} x_u x_v$, i.e., the polynomial corresponding to M_1. Clearly, if the bichromatic edges in M_1 are distributed evenly across all sets in the partition $B_{11}, \ldots, B_{1k/2}$, $\mathsf{rank}_\varphi(h_1) = (((1+\epsilon)/3k)n)^{k/2}$. However, this is not possible in general. Nevertheless, we get a smaller but good enough bound by a simple averaging argument. Let $\mathsf{be}_\varphi(M_i) = \sum_{j\in[\frac{k}{2}]} \mathsf{be}_\varphi(B_{ij})$. We have $\mathsf{be}_\varphi(M_1) \geq \frac{(1+\epsilon)n}{6}$. Let $\alpha = |\{j \mid \mathsf{be}_\varphi(B_{1,j}) \geq n/20k\}|$. Then

$$\frac{(1+\epsilon)n}{6} \leq \mathsf{be}_\varphi(M_1) \leq \alpha\frac{n}{k} + (k/2-\alpha)\frac{n}{20k}$$
$$\text{i.e., } \frac{(1+\epsilon)n}{6} \leq \alpha\frac{n}{k} + (k/2-\alpha)\frac{n}{20k}$$
$$\implies \alpha \geq \frac{(23+20\epsilon)}{114}k.$$

Note that $\mathsf{rank}_\varphi(\sum_{(u,v)\in B_{1j}} x_u x_v) = \mathsf{be}_\varphi(B_{1j})$ and hence we have $\mathsf{rank}_\varphi(h_1) \geq (\frac{n}{20k})^\alpha = \left(\frac{n}{k}\right)^{ck}$ for some constant $c > 0$ as required. □

4 Lower bounds

In this section we prove parameterized lower bounds for some special classes of syntactic multilinear ABPs. In particular, we prove lower bounds for the size of ROABPs, strict interval ABPs and a sum of restricted class of ROPs.

4.1 ROABP

In this section we prove a parameterized lower bound for the size of any ROABP computing the polynomials defined in Section 3. The lower bound argument follows from the fact that for any polynomial computed by an ROABP P, there exists an equi-partition φ of variables such that $\mathsf{rank}_\varphi(P)$ is bounded by the size of the ROABP [19].

Theorem 3. *Any ROABP computing the polynomial family $f = (f_{n,2k})$ requires size $\Omega(n^k/(2k)^{2k})$.*

Proof. Let P be an ROABP of size S computing f. Consider an ordering from left to right of the variables occurring in the ROABP, $x_1, x_2, \ldots, x_n$. We can define the equi-partition $\varphi : X \to Y \cup Z$ such that,

$$\varphi(x_i) = \begin{cases} y_i, & \text{if } i \leq n/2 \\ z_{i-n/2} & \text{otherwise.} \end{cases}$$

Now, let L_i be a layer in P such that incoming edges to L_i are labelled with a linear polynomial in x_i. Then, we can represent f as

$$f(x_1, \ldots, x_n) = \sum_{j\in L_{n/2}} [s, v_j]_P \cdot [v_j, t]_P.$$

By definition of φ, for all $v_j \in L_{n/2}$, $\mathsf{rank}_\varphi([s, v_j]_P \cdot [v_j, t]_P) = 1$.

Then, $\mathsf{rank}_\varphi(f) \leq |L_{n/2}| \leq S$. By Theorem 1, $\mathsf{rank}_\varphi(f) = \Omega(n^k/(2k)^{2k})$, therefore we have $S = \Omega(n^k/(2k)^{2k})$ as required. □

Combining Theorem 3 with Theorem 2 we get:

Theorem 4. *An ROABP computing the family of polynomials h defined in Section 3 required size $n^{\Omega(k)}$.*

Proof. Follows from the proof of Theorem 3 that for any size S ROABP computing the polynomial h, there is an equi-partition φ such that $\mathsf{rank}_\varphi(h) \leq S$. Then by Theorem 2, we have $S = n^{\Omega(k)}$ as required. □

4.2 Strict interval ABPs

In this section we prove a parameterized lower bound against the polynomial family f defined in Section 3 for the size of strict interval ABPs. Without loss of generality, assume that π is the identity permutation. Let P be a π strict-interval ABP computing the polynomial f. As a crucial ingredient in the lower bound proof, we show that using the standard divide and conquer approach, a strict-interval ABP can be transformed into a depth four circuit with $n^{\sqrt{k}}$ blow up in the size. To begin with, we need the following simple depth reduction for strict interval ABPs computing degree k polynomials. Proof is omitted.

Lemma 4. *Let P be a syntactic multilinear ABP of size S computing a homogeneous degree k polynomial g on n variables. Then there is a syntactic multilinear ABP P' of depth $k+1$ and size $O(S \cdot k)$ computing g such that:*

1. *Every node in the i^{th} layer of P' computes a homogeneous degree i polynomial.*
2. *If P is strict interval then so is P'.*

Using Lemma 4 we can obtain a parameterized version of depth reduction to depth four circuits:

Lemma 5. *Let $g(x_1, \ldots, x_n)$ be a multilinear polynomial of degree k computed by a syntactic multilinear branching program P of size S. Then*

$$g(x_1, \ldots, x_n) = \sum_{i=1}^{T} \prod_{j=1}^{\sqrt{k}} f_{i,j} \tag{1}$$

for some $T = S^{O(\sqrt{k})}$ and $f_{i,j}$ is a degree $\sqrt{k}$ multilinear polynomial computed by a sub-program of P for $i \in \{1, \ldots, T\}, j \in \{1, \ldots, \sqrt{k}\}$.

Now, to prove the claimed lower bound for the size of strict interval ABPs, all we need is given a polynomial f computed by an strict interval ABP of size S, an equi-partition φ of X such that $\mathsf{rank}_\varphi(f) << n^k$.

Lemma 6. *Let f be a polynomial computed by a strict interval ABP of size S. Then there is a partition φ such that $\mathsf{rank}_\varphi(f) \leq S^{O(\sqrt{k})} n^{\sqrt{k}}$.*

Proof. Without loss of generality, assume that P is a strict interval ABP with respect to the identity permutation. Let $\varphi_{\mathsf{mid}} : X \to Y \cup Z$ be the partition

$$\varphi_{\mathsf{mid}}(x_i) = \begin{cases} y_i, & \text{if } i \leq n/2, \\ z_{i-n/2} & \text{otherwise.} \end{cases}$$

Consider the representation for f as in (1). Then for every $1 \leq i \leq T$, for all but one j, we have either $\varphi_{\mathsf{mid}}(\mathsf{var}([i_j, i_{j+1}])) \subseteq Y$ or $\varphi_{\mathsf{mid}}(\mathsf{var}([i_j, i_{j+1}])) \subseteq Z$. Therefore, $\mathsf{rank}_{\varphi_{\mathsf{mid}}}([s, i_1]_P \cdot \prod_{m=1}^{\sqrt{k}-2} [i_m, i_{m+1}]_P \cdot [i_{\sqrt{k}-1}, t]_P) \leq n^{\sqrt{k}}$, for every $i_j \in L_{j\sqrt{k}}$. By sub-additivity of rank_φ, we have $\mathsf{rank}_\varphi(f) \leq S^{O(\sqrt{k})} n^{\sqrt{k}}$ for $\varphi = \varphi_{\mathsf{mid}}$. □

The required lower bound is immediate now.

Corollary 1. *Any strict-interval ABP computing the polynomial f has size $n^{\Omega(\sqrt{k})}$.*

4.3 Rank bound for ROPs by Graph representation

The reader might be tempted to believe that the lower bound arguments in the preceding sections might be applicable to more general models such as sum of ROFs and sum of ROABPs or even multilinear formulas. However, as we have seen in Section 3, there is a sum of three ROFs that has high rank under every partition. Thus our approach using rank_φ as a complexity measure is unlikely to yield lower bounds for even sum of ROFs, which is in contrast to the classical setting, where exponential lower bounds against models such as sum of ROFs and sum of ROABPs follow easily.

In this section, we develop a new method of analyzing rank of degree k polynomials computed by ROFs. Let $p \in \mathbb{F}[X]$ be the polynomial computed by a ROF Φ. We want to construct a graph $G_p = (X, E_p)$ corresponding to p so that $\mathsf{rank}_\varphi(p)$ can be related to certain parameters of the graph. A v in Φ is said to be a *maximal-degree-two gate* if v computes a degree two polynomial, and the parent of v computes a polynomial whose degree is strictly greater than two. Further, v is said to be a *maximal-degree-one* gate if v computes a linear form and the parent of v computes a polynomial of degree strictly greater than one. A gate v at depth 1 is said to be a *high degree gate* if the degree of the polynomial computed at v is strictly greater than two. Let V_2 denote the set of all maximal-degree-two gates in Φ, V_1 denote the set of all maximal-degree-one gates and V_0 denote the set of all high degree gates in Φ at depth one. Let $\mathsf{atomic}(\Phi) = V_0 \cup V_1 \cup V_2$. The following is a straightforward observation:

Observation 1 *Let Φ be an ROF and v be a maximal-degree-two gate in Φ. Then the polynomial Φ_v computed is of the form $\Phi_v = \sum_{i=1}^{s} \ell_{i_1}\ell_{i_2}$, where ℓ_{i_j} $1 \le i \le s$, $j \in \{1, 2\}$ are variable disjoint linear forms for some $s > 0$ such that each of the ℓ_{i_j} is dependent on at least one variable.*

For a linear form $\ell = \sum_{j=1}^{r} \alpha_{i_j} x_{i_j}$, let $\mathsf{path}(\ell)$ be the simple undirected path (x_{i_1}, x_{i_2}), $(x_{i_2}, x_{i_3}), \ldots, (x_{i_{r-1}}, x_{i_r})$. In the case when $r = 1$, $\mathsf{path}(\ell)$ is just single vertex. Similarly, for a subset $S \subseteq X$ of variables, let $\mathsf{path}(S)$ denote the path (x_{i_1}, x_{i_2}), $(x_{i_2}, x_{i_3}), \ldots,$ $(x_{i_{r-1}}, x_{i_r})$ where $S = \{x_{i_1}, \ldots, x_{i_r}\}$, $i_1 < i_2 < \ldots < i_r$. For two variable disjoint linear forms ℓ and ℓ', let $\mathsf{path}(\ell\ell')$ be the path obtained by connecting the last vertex in $\mathsf{path}(\ell)$ to the first vertex of $\mathsf{path}(\ell')$ by a new edge. Now, we define a graph $G_p = (X, E_p)$ where vertices correspond to variables $x_u \in X$ and the set of edges E_p defined as follows. For each $v \in \mathsf{atomic}(\Phi)$ we add the following edges to E_p:

Case 1 $\Phi_v = \sum_{i=1}^{r} \ell_{i_1}\ell_{i_2}$ for some $r > 0$ add $\mathsf{path}(\ell_{i_1}\ell_{i_2})$ to G_p for every $1 \le i \le t$.
Case 2 $\Phi_v = \prod_{i\in S} x_i$ or $\Phi_v = \sum_{i\in S} c_i x_i$, where $S \subseteq X$, c_is are constants from $\mathbb{F}$, add $\mathsf{path}(S)$ to G_p.

It may be noted that the graph G_p is not unique as it depends on the given minimal ROF Φ computing f. In the following, we show that for a given partition φ, we bound the $\mathsf{rank}_\varphi(p)$ in terms of the number of bichromatic edges $\mathsf{be}_\varphi(G_p)$. We have:

Theorem 5. *Let $p \in \mathbb{F}[X_1, \ldots, X_n]$ be a multilinear polynomial of degree k computed by a ROF Φ. Then, for any any equi-partition $\varphi : X \to Y \cup Z$, $\mathsf{rank}_\varphi(p) \le (4\mathsf{be}_\varphi(G_p))^{\frac{k}{2}}$.*

Proof. The proof is by induction on the structure of Φ. The base case is when the root gate of Φ is in $\mathsf{atomic}(\Phi)$. Consider a node $v \in \mathsf{atomic}(\Phi)$.

Case 1 $\Phi_v = \sum_{(i,j)\in S} x_i x_j$. If $\varphi(x_i)$, $\varphi(x_j)$ are not in the same partition, then each monomial $x_i x_j$ contributes 1 towards $\mathsf{rank}_\varphi(p)$. At the same time, the edge (x_i, x_j) added to E_p is bichromatic, so each monomial contributes 1 towards the measure $\mathsf{be}_\varphi(G_p)$ as well.

Case 2 $\Phi_v = \sum_{(a,b)\in T} \ell_a \ell_b$. If, for some $x_i, x_j \in \mathsf{var}(\ell_a)$, $\varphi(x_i), \varphi(x_j)$ are in different partitions, then the linear form ℓ_a contributes 2 towards $\mathsf{rank}_\varphi(\ell_a)$. If the same holds true for ℓ_b, then $\ell_a\ell_b$ would together contribute 4 towards $\mathsf{rank}_\varphi(p)$ and ≥ 2 towards the measure $\mathsf{be}_\varphi(G_p)$.

Case 3 $\Phi_v = \sum_{i\in W_1} c_i x_i$ or $\Phi_v = \prod_{i\in W_2} x_i$ for some $W_1, W_2 \subseteq X$. The first case has been considered already. For the second case, if $\exists x_a, x_b \in W_2$ such that $\varphi(x_a), \varphi(x_b)$ are in different partitions, the polynomial computed by the gate v will contribute a 1 towards $\mathsf{rank}_\varphi(p)$ and at least 1 towards $\mathsf{be}_\varphi(G_p)$, otherwise it contributes 0.

Thus we have verified that the statement is true when the root gate v of Φ is contained in $\mathsf{atomic}(\Phi)$. Suppose $p = p_1 \mathsf{\ op\ } p_2$ for $\mathsf{op} \in \{+, \times\}$ where p_1 and p_2 are variable disjoint and are computed by ROFs. By induction hypothesis, $\mathsf{rank}_\varphi(p_j) \leq (4\mathsf{be}_\varphi(G_{p_j}))^{\frac{k_j}{2}}$ where $k_j = \deg(f_j)$. As $\mathsf{be}_\varphi(G_p) = \mathsf{be}_\varphi(G_{p_1}) + \mathsf{be}_\varphi(G_{p_2})$ and $k = k_1 + k_2$ ($\mathsf{op} = \times$) or $k = \max\{k_1, k_2\}$ ($\mathsf{op} = +$) we have, $\mathsf{rank}_\varphi(f) \leq (4\mathsf{be}_\varphi(G_p))^{\frac{k}{2}}$ as required. □

Recall that bisection of an undirected graph $G = (V, E)$ is a set $S \subseteq V$ such that $|S| = |V|/2$. The size of a bisection S is the number of edges across S and $\overline{S}$, i.e., $|\{(u, v) \mid (u, v) \in E, u \in S, v \notin S\}|$. The following is an immediate corollary to Theorem 5:

Theorem 6. *Let G be a graph on n vertices such that there is a bisection of G of size $n^{1-\epsilon}$. Suppose $p_1, \ldots, p_s$ be ROFs such that G_{p_i} is a sub-graph of G. Then, if $p = p_1 + \cdots + p_S$ we have $S = (n^{\Omega(k)}/t(k))$, where t is a computable function on k.*

Proof. Let $C = (S, \overline{S})$ be the cut and $\mathsf{size}(C)$ denote the number of edges across the cut. Define a partition $\varphi : X \to Y \cup Z$ as follows:

$$\varphi(x_i) \in \begin{cases} Y & \text{if } i \in S, \\ Z & \text{otherwise.} \end{cases}$$

Then by Theorem 5, $\mathsf{rank}_\varphi(p_i) \leq \mathsf{be}_\varphi(G_{p_i}))^{\frac{k}{2}}$. Since G_{p_i} is a sub-graph of G, we have $\mathsf{be}_\varphi(G_p)) \leq \mathsf{size}(C) \leq n^{1-\epsilon}$. Therefore, $\mathsf{rank}_\varphi(p_i) \leq O_k(n^{(1-\epsilon)k/2})$. By sub-additivity, we have $\mathsf{rank}_\varphi(f) \leq SO_k(n^{(1-\epsilon)k/2})$ where O_k is upto a factor that depends only on a function of k. By Theorem 1, we get $S = \Omega(n^{\epsilon k/2})$. □

Conclusions

Our results demonstrate the challenges in translating classical arithmetic circuit lower bounds to the parameterized setting, when the degree of the polynomial is the parameter. We get a full rank polynomial that can be computed by depth four arithmetic circuits of fpt size, whereas in the classical setting, full rank polynomials cannot be computed by multilinear formulas of polynomial size [21]. This makes the task of proving parameterized lower bounds for algebraic computation much more challenging task. Given the application of polynomials whose degree is bound by a parameter in the design of efficient parameterized algorithms for many counting problems, we believe that this is a worthy research direction to pursue.

Further, we believe that our results are an indication that study of parameterized complexity of polynomials with degree as the parameter could possibly shed more light on the use of algebraic techniques in parameterized algorithms.

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